# Free Martingale polynomials for stationary Jacobi processes

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# FREE MARTINGALE POLYNOMIALS FOR STATIONARY JACOBI PROCESSES

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Abstract. We generalize a previous result concerning free martingale polynomials for the stationary free Jacobi process of parameters $\lambda \in ]0.1], \theta = 1/2$. Hopelessly, apart from the case $\lambda = 1$, the polynomials we derive are no longer orthogonal with respect to the spectral measure. As a matter of fact, we use the multiplicative renormalization method to write down its corresponding orthogonal polynomials as well as the orthogonality measure associated with the martingale polynomials. We finally give a realization of the spectral measure of the free stationary Jacobi process by means of the corresponding one mode interacting Fock space.

## 1. Preliminaries

Let $(\mathscr{A}, \phi)$ a $W^\star$-non commutative probability space. Easily speaking, $\mathscr{A}$ is a unital von Neumann algebra and $\phi$ is a tracial faithful linear functional (state). In a previous work ([8]), we defined, via matrix theory, and studied a two parameters-dependent self-adjoint free process, called free Jacobi process. Our focus will be on a particular case called the stationary Jacobi process since its spectral distribution does not depend on time. It is defined as $J_t := PUY_tQY_t^\star U^\star P$ where

- $(Y_t)_{t\geq 0}$ is a free multiplicative Brownian motion (see [7]).
- $U$ is a Haar unitary operator in $(\mathscr{A}, \Phi)$.
- $P$ is a projection with $\Phi(P) = \lambda\theta \leq 1, \quad \theta \in ]0,1]$.
- $Q$ is a projection with $\Phi(Q) = \theta$.
- $QP = PQ = \begin{cases} P & \text{if} \quad \lambda \leq 1 \\ Q & \text{if} \quad \lambda > 1 \end{cases}$
- $\{U, U^\star\}$ and $\{P, Q\}$ are free (see [12] for freeness).

Thus the process takes values in *the compressed space* $(P\mathscr{A}P, (1/\phi(P))\phi)$. The spectral distribution has the following decomposition :

$$\mu_{\lambda,\theta}(dx) = \frac{1}{2\pi\lambda\theta}\frac{\sqrt{(x_+ - x)(x - x_-)}}{x(1-x)}\mathbf{1}_{[x_-,x_+]}(x)dx + a_0\delta_0(dx) + a_1\delta_1(dx)$$

where $\delta_y$ stands for the Dirac mass at $y$ with corresponding weight $a_y$, $y \in \{0,1\}$ and

$$x_\pm = \left(\sqrt{\theta(1-\lambda\theta)} \pm \sqrt{\lambda\theta(1-\theta)}\right)^2$$

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Its Cauchy transform writes

$$G_{\mu_{\lambda,\theta}}(z) = \frac{(2-(1/\lambda\theta))z + (1/\lambda - 1) + \sqrt{Az^2 - Bz + C}}{2z(z-1)}, \; z \in \mathbb{C} \setminus [0,1] \tag{1}$$

with $A = 1/(\lambda\theta)^2$, $B = 2((1/\lambda\theta)(1+1/\lambda) - 2/\lambda)$ et $C = (1-1/\lambda)^2$. It was shown in [8] that if $\lambda \in ]0,1], 1/\theta \geq \lambda + 1$ then the process is injective in $P\mathscr{A}P$, that is $a_0 = a_1 = 0$. Moreover, $\mu_{1,1/2}(dx)$ fits the Beta distribution $B(1/2,1/2)$:

$$\mu_{1,1/2}(dx) = \frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)dx$$

Recall that the Tchebycheff polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x), \; n \geq 0, \; |x| \leq 1.$$

and that they are orthogonal with respect to $\mu_{1,1/2}(dx)$. Their generating function is given by:

$$g(u,x) = \sum_{n\geq 0} T_n(x)u^n = \frac{1-ux}{1-2ux+u^2}, \quad |u| < 1.$$

In [8], we proved that for $r > 0$

$$g(re^t, J_t) = ((1+re^t)P - 2e^t J_t)((1+re^t)^2 P - 4re^t J_t)^{-1}, \quad t < -\ln r$$

defines a free martingale with respect to the natural filtration of $J$, say $\mathscr{J}_t$, the unit of the compressed space being the projection $P$. It follows that $(e^{nt}T_n(2J_t - P))_{t\geq 0}$, $n \geq 1$ is a family of free martingale polynomials. Note also that

$$\begin{aligned} h(re^t, J_t) := 2g(re^t, J_t) - P &= \frac{(1-r^2e^{2t})}{(1+re^t)^2}(P - \frac{4re^t}{(1+re^t)^2}J_t)^{-1} \\ &= \frac{1-re^t}{1+re^t}(P - \frac{4re^t}{(1+re^t)^2}J_t)^{-1} \\ &= (1-(re^t)^2)(P - 2re^t(2J_t - P) + (re^t)^2)^{-1} \end{aligned}$$

is also a free martingale. Let $U_n$ denote the $n$-th Tchebycheff polynomial of the second kind defined by

$$U_n(\cos\alpha) = \frac{\sin(n+1)\alpha}{\sin\alpha}, \quad \alpha \in \mathbb{R}$$

with generating function given by

$$\sum_{n\geq 0} U_n(x)u^n = \frac{1}{1-2ux+u^2}, \quad |x| \leq 1, \; |u| < 1.$$

Then, one deduces either from the above generating function or from the relation $2T_n = U_n - U_{n-2}, U_{-1} := 0$ that $\{M_t^n := e^{nt}(U_n - U_{n-2})(2J_t - P), \; n \geq 1\}_{t\geq 0}$ is a family of free martingale polynomials. The aim of this work is to extend this claim to the range $\theta = 1/2, \lambda \in ]0,1]$. The motivation originates from [10] where the author determines the family of orthogonal polynomials with respect to $\mu_{\lambda,\theta}$. Our first guess was that these will be free martingales polynomials for all $\lambda \in ]0,1]$, $\theta \leq 1/(\lambda+1)$. Yet, things turn to be more complicated: not only the range is restricted but the martingale polynomials we derive are not orthogonal with respect to $\mu_{\lambda,1/2}$ except for $\lambda = 1$. As a matter of fact, we will on one hand derive the orthogonal polynomials corresponding to $\mu_{\lambda,1/2}$ and compute on the other hand the appropriate orthogonality measure for our martingales polynomials. The last part of the paper is devoted to a realization of the free stationary Jacobi process using the Accardi-Bozejko isomorphism (see [1]) as well as some comments.

*Remark.* From a matrix theory point of view, the choice $\theta = 1/2$ correponds to the ultra-spherical multivariate Beta distribution (see [8]). Moreover, to our level of Knowledge, there is only one result concerning martingale polynomials for the stationary (classical) Jacobi process, which is restricted to the one dimensional case. More precisely, pick a vector $(x_1, \ldots, x_d)$ belonging to the sphere $S^{d-1}, d \geq 2$ distributed according to the uniform (Haar) measure, then form the discrete process defined by

$$s_p = \sum_{i=0}^{p} x_i^2, \quad 1 \leq p \leq d-1.$$

It is a known that each random variable has the Beta distribution $B((d-p/2), p/2)$. It was shown in [11] that

$$M_n^d(p) = \frac{1}{((d-p)/2)_n} P_n^{\alpha,\beta}(2s_p - 1),$$

where $P_n^{\alpha,\beta}$ denotes the $n$-th Jacobi polynomial of parameters $\alpha = (d-p)/2 - 1, \beta = (p/2) - 1$, is a martingale with respect to the natural filtration of the process. To relate this to our work, we rewrite $s_p$ in the matrix form

$$s_p = P_1 U_d Q_p U_d^{\star} P_1,$$

where $U_d$ is a $d \times d$ Haar unitary matrix, $P_1$ is a $d \times d$ projection with only one non vanishing coefficient $(P_1)_{11} = 1$ and $Q_p$ is a $d \times d$ projection with only $p$ non vanishing term $(Q_p)_{11} = \cdots = (Q_p)_{pp} = 1$. For $d = 2p$, we get the ultraspherical polynomials of parameter $\lambda = (p-1)/2$.

## 2. Main result

One has for $\lambda \in ]0,1]$, $\theta = 1/2$

$$x_- = \left(\frac{\sqrt{2-\lambda}}{2} - \frac{\sqrt{\lambda}}{2}\right)^2 \leq x \leq x_+ = \left(\frac{\sqrt{2-\lambda}}{2} + \frac{\sqrt{\lambda}}{2}\right)^2 \Rightarrow -1 \leq \frac{2x-1}{\sqrt{\lambda(2-\lambda)}} \leq 1$$

and our main result is stated as follows:

**Proposition 2.1.** *Set*

$$a(\lambda) = \frac{(1-\lambda)}{\sqrt{\lambda(2-\lambda)}}, \; x_{t,\lambda} = \frac{2J_t - P}{\sqrt{\lambda(2-\lambda)}}$$

*For each $n \geq 1$, the process defined by*

$$M_t^n := [U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})] \left(\frac{e^t}{\lambda(2-\lambda)}\right)^n, t \geq 0$$

*is a $(\mathscr{J}_t)$-free martingale.*

## 3. Proof of the main result

The proof consists of two parts: the first one consists in deriving a martingale function for all values of $\lambda \in ]0,1], \theta \leq 1/2 \leq 1/(\lambda+1)$. In the second one, we specialize for $\theta = 1/2$ and show that this function correponds to the generating function of the polynomials stated above.

*First step*: inspired by the above expression of $h(re^t, J_t)$, we will look for martingales of the form

$$R_t := K_t(P - Z_tJ_t)^{-1} = K_t \sum_{n\geq 0} Z_t^n J_t^n := K_tH_t$$

where $K, Z$ are differentiable functions of the variable $t$ lying in some interval $[0, t_0[$ such that $0 < Z_t < 1$ for $t \in [0, t_0[$. The finite variation part of $dR_t$ is given by

$$FV(dR_t) = K_t' H_t dt + K_t FV(dH_t)$$

Our main tool is the free stochastic calculus and more precisely the free stochastic differential equation already set for $J_t^n$, $n \geq 1$ ([8]):

$$dJ_t^n = dM_t + n(\theta P - J_t)J_t^{n-1}dt + \lambda\theta \sum_{l=1}^{n-1} l[m_{n-l}(P - J_t)J_t^{l-1} + (m_{n-l-1} - m_{n-l})J_t^l)]dt$$

where $dM$ stands for the martingale part and $m_n$ is the $n$-th moment of $J_t$ in $P\mathscr{A}P$:

$$m_n := \tilde{\phi}(J_t^n) := \frac{1}{\phi(P)}\phi(J_t^n)$$

The finite variation part $FV(dJ_t^n)$ of $J_t^n$ transforms to:

$$\begin{aligned}
FV(dJ_t^n) &= n(\theta P - J_t)J_t^{n-1}dt + \lambda\theta \left[\sum_{l=1}^{n-1} l[m_{n-l}J_t^{l-1} + \sum_{l=1}^{n-1} l(m_{n-l-1} - 2m_{n-l})J_t^l)\right] dt \\
&= n(\theta P - J_t)J_t^{n-1}dt + \lambda\theta \sum_{l=1}^{n-1} l m_{n-l}J_t^{l-1} + \sum_{l=1}^{n}(l-1)(m_{n-l} - 2m_{n-l+1})J_t^{l-1}dt \\
&= n(\theta P - J_t)J_t^{n-1}dt + \lambda\theta \sum_{l=1}^{n}[l m_{n-l} + (l-1)(m_{n-l} - 2m_{n-l+1})]J_t^{l-1}dt - n\lambda\theta J_t^{n-1}dt \\
&= n\theta(1-\lambda)J_t^{n-1}dt - nJ_t^n dt + \lambda\theta \sum_{l=1}^{n}[m_{n-l} + 2(l-1)(m_{n-l} - m_{n-l+1})]J_t^{l-1}dt
\end{aligned}$$

Thus

$$
\begin{aligned}
FV(dH_t) &= \sum_{n\geq 1} nZ_t'Z_t^{n-1}J_t^n dt + \sum_{n\geq 1} Z_t FV(J_t^n) \\
&= \sum_{n\geq 1} nZ_t'Z_t^{n-1}J_t^n dt - \sum_{n\geq 0} nZ_t^n J_t^n dt + \theta(1-\lambda)\sum_{n\geq 1} nZ_t^n J_t^{n-1}dt \\
&+ \lambda\theta\sum_{n\geq 1}\sum_{l=1}^{n} Z_t^n m_{n-l}J_t^{l-1}dt + 2\lambda\theta\sum_{n\geq 1}\sum_{l=1}^{n}(l-1)Z_t^n(m_{n-l}-m_{n-l+1})]J_t^{l-1}dt \\
&= \sum_{n\geq 1} n[Z_t'Z_t^{n-1} - Z_t^n]J_t^n dt + \theta(1-\lambda)\sum_{n\geq 0}(n+1)Z_t^{n+1}J_t^n dt \\
&+ \lambda\theta\sum_{n\geq 0}\sum_{l\geq 0} Z_t^{n+l+1}m_n J_t^l dt + 2\lambda\theta\sum_{n\geq 0}\sum_{l\geq 0} lZ_t^{n+l+1}(m_n - m_{n+1})]J_t^l dt
\end{aligned}
$$

$$
\begin{aligned}
&= [Z_t'/Z_t - 1 + \theta(1-\lambda)Z_t]\sum_{n\geq 1} nZ_t^n J_t^n dt + \theta(1-\lambda)Z_t\sum_{n\geq 0} Z_t^n J_t^n dt \\
&+ \lambda\theta\sum_{n\geq 0} Z_t^{n+1}m_n \sum_{l\geq 0} Z_t^l J_t^l dt + 2\lambda\theta\sum_{n\geq 0} Z_t^{n+1}(m_n - m_{n+1})]\sum_{l\geq 0} lZ_t^l J_t^l dt
\end{aligned}
$$

Recall that the Cauchy transform of a measure on the real line is defined by

$$G_\nu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\nu(dx) = \sum_{n\geq 0}\frac{1}{z^{n+1}}\int_{\mathbb{R}} x^n \nu(dx)$$

for some values of $z$ for which both the integral and the infinite sum make sense. Then, since $0 < Z < 1$ and $\mu_{\lambda,\theta}$ is supported in $[0,1]$, it is easy to see that

$$\sum_{n\geq 0} Z_t^{n+1}(m_n - m_{n+1}) = \left(1 - \frac{1}{Z_t}\right) G_{\mu_{\lambda,\theta}}\left(\frac{1}{Z_t}\right) + 1$$

with $G_{\mu_{\lambda,\theta}}$ given by (1). This gives

$$2\lambda\theta(1-z)G_{\mu_{\lambda,\theta}}(z) = \frac{(1-2\lambda\theta)z - \theta(1-\lambda) - \sqrt{z^2 - (\lambda\theta)^2 Bz + (\lambda\theta)^2 C}}{z},$$

so that

$$2\lambda\theta(1-Z_t^{-1})G_{\mu_{\lambda,\theta}}(Z_t^{-1}) + 2\lambda\theta = 1 - \theta(1-\lambda)Z_t - \sqrt{1-(\lambda\theta)^2 BZ_t + (\lambda\theta)^2 CZ_t},$$

We finally get:

$$
\begin{aligned}
FV(dH_t) &= [Z_t'/Z_t - \sqrt{1-(\lambda\theta)^2 BZ_t + (\lambda\theta)^2 CZ_t^2}]\sum_{n\geq 1} nZ_t^n J_t^n dt \\
&+ \left[\lambda\theta G_{\mu_{\lambda,\theta}}\left(\frac{1}{Z_t}\right) + \theta(1-\lambda)Z_t\right]\sum_{n\geq 0} Z_t^n J_t^n dt
\end{aligned}
$$

In order to derive free martingales, we shall pick $Z$ such that $Z_t' = Z_t\sqrt{1-(\lambda\theta)^2 BZ_t + (\lambda\theta)^2 CZ_t^2}$. This shows that $Z$ is an increasing function and one can solve the above non linear differential equation as follows: use the variables change $u = Z_t$, $t < t_0$, then integrate to get :

$$\int_{[Z_0,Z_t]} \frac{du}{u\sqrt{1-2\theta(1+\lambda-2\lambda\theta)u+(\theta(1-\lambda))^2u^2}} = t$$

*Remark.* Let $c_1 = 2\theta(1+\lambda-2\lambda\theta), c_2 = \theta^2(1-\lambda)^2$. Then, the function $u \mapsto 1-c_1u+c_2u^2$ is decreasing for $u \in ]0,1[$: in fact,

$$\begin{aligned} 2c_2u - c_1 < 2c_2 - c_1 &= 2\theta^2(1-\lambda)^2 - 2\theta(1+\lambda-2\lambda\theta) \\ &= 2\theta[\theta(1+\lambda^2)-(1+\lambda)] \leq 2\theta\left(\frac{1+\lambda^2}{1+\lambda}-(1+\lambda)\right) = -\frac{4\lambda\theta}{1+\lambda} < 0 \end{aligned}$$

which yields $1-c_1u+c_2u^2 > 1-c_1+c_2 = (1-\theta(1+\lambda))^2 \geq 0$.

Next, use the variable change $1-vu = \sqrt{1-c_1u+c_2u^2}$. This gives

$$u = \frac{2v-c_1}{v^2-c_2},\ du = -2\frac{v^2+c_2-c_1v}{(v^2-c_2)^2}dv,\ 1-vu = -\frac{v^2+c_2-c_1v}{v^2-c_2}$$

Moreover

$$u \mapsto v = \frac{1-\sqrt{1-c_1u+c_2u^2}}{u}, \quad 0<u<1$$

is an increasing function: in fact the numerator of its derivative writes

$$c_1u - 2c_2u^2 + 2(1-c_1u+c_2u^2) - 2\sqrt{1-c_1u+c_2u^2} = (2-c_1u) - 2\sqrt{1-c_1u+c_2u^2}$$

Since $2-c_1u > 2-c_1 = 2(1-\theta(1+\lambda)) + 4\lambda\theta^2 > 0$, our claim follows from the fact that $c_1^2 - 4c_2 = 16\lambda\theta^2(1-\lambda\theta)(1-2\theta) \geq 0$.

Finally, the integral transforms to

$$\int_{[v_0,v_t]} \frac{2dv}{2v-c_2} = \log\left|\frac{2v_t-c_1}{2v_0-c_1}\right| = t$$

where $1-Z_tv_t = \sqrt{1-c_1Z_t+c_2Z_t^2}$, $1-Z_0v_0 = \sqrt{1-c_1Z_0+c_2Z_0^2}$. Note also that $c_1^2-4c_2 \geq 0$ implies that for all $u \in [Z_0,Z_t] \subset ]0,1[$

$$\begin{aligned} v - \frac{c_1}{2} &= \frac{1-\sqrt{1-c_1u+c_2u^2}}{u} - \frac{c_1}{2} = \frac{(1-c_1u/2)-\sqrt{1-c_1u+c_2u^2}}{u} \\ &= \frac{(1-c_1u/2)^2-(1-c_1u+c_2u^2)}{u((1-c_1u/2)+\sqrt{1-c_1u+c_2u^2})} \geq 0 \end{aligned}$$

since $1-c_1/2u \geq 1-c_1/2 \geq 0$. Thus $v \geq c_1/2 \geq \sqrt{c_2}$.

$$v_t = [(2v_0-c_1)e^t+c_1]/2 \Leftrightarrow \sqrt{1-c_1Z_t+c_2Z_t^2} = 1 - \frac{(2v_0-c_1)e^{\pm t}+c_1}{2}Z_t$$

We finally get

$$Z_t = \frac{4(2v_0-c_1)e^{\pm t}}{((2v_0-c_1)e^t+c_1)^2-4c_2}, \quad t \leq t_0$$

where $t_0$ is the first time such that $Z_{t_0} = 1 \Leftrightarrow (2v_0-c_1)e^{t_0}+c_1)^2 - 4c_2 - 4(2v_0-c_1)e^{t_0}$. Set $r = r(\lambda,\theta) := (2v_0-c_1)$ and $x_0 = e^{t_0} > 1$, then $r^2x_0^2 + 2(c_1-2)rx_0 + c_1^2 - 4c_2 = 0$. The discriminant equals to $\Delta = 16r^2(1+c_2-c_1) = 16r^2(1-\theta(1+\lambda))^2$. Thus

$$x_0 = \frac{-(c_1-2)-2(1-\theta(1+\lambda))}{r} = \frac{2(1-\theta(1+\lambda))+4\lambda\theta^2-2(1-\theta(1+\lambda))}{r} = \frac{4\lambda\theta^2}{r} \geq 1$$

The last inequality follows from the fact that $1-\sqrt{c_2}u \geq 1-\theta(1+\lambda) \geq 0$ and from

$$r - 4\lambda\theta^2 = 2v_0 - c_1 - 4\lambda\theta^2 = 2(v_0 - \theta(1+\lambda)) = 2(v_0 - \sqrt{c_2}) \leq 0.$$

It gives $t_0 = -\ln(r/4\lambda\theta^2)$. Note also that the denominator is well defined for all $t \leq t_0$ since $c_1^2 \geq 4c_2$ and $2v_0 - c_1 \geq 0$.
For the ramaining terms, we shall choose $K$ such that

$$K_t' + K_t\left[\lambda\theta G_{\mu_{\lambda,\theta}}\left(\frac{1}{Z_t}\right) + \theta(1-\lambda)Z_t\right] = 0$$

An easy computation shows that this equals to

$$K_t' + \frac{K_t}{2}\left[\theta(1-\lambda)\frac{Z_t^2}{Z_t-1} + (1-2\theta)\frac{Z_t}{Z_t-1} - \frac{Z_t\sqrt{1-c_1Z_t+c_2Z_t^2}}{Z_t-1}\right] = 0$$

Remembering the choice of the function $Z$, this writes

$$K_t' - \frac{K_t}{2}\left[\frac{Z_t'}{Z_t-1} - (1-2\theta)\frac{Z_t}{Z_t-1} - \theta(1-\lambda)\frac{Z_t^2}{Z_t-1}\right] = 0$$

or equialently

$$K_t' - \frac{K_t}{2}\left[\frac{Z_t'}{Z_t-1} - (1-\theta-\lambda\theta)\frac{Z_t}{Z_t-1} - \theta(1-\lambda)Z_t\right] = 0$$

If $K_t \neq 0$, then

$$\log K_t = \frac{1}{2}\log(1-Z_t) - \frac{1-\theta-\lambda\theta}{2}\int\frac{Z_s}{Z_s-1}ds - \frac{\theta(1-\lambda)}{2}\int Z_s ds + C$$

If $\lambda \neq 1$, then the last term is given by

$$-\frac{\theta(1-\lambda)}{2}\int Z_s ds = \frac{\theta(1-\lambda)}{\sqrt{c_2}}\int\frac{(r/2\sqrt{c_2})e^t}{1-\left(\dfrac{re^t+c_1}{2\sqrt{c_2}}\right)^2} = \arg\tanh\left(\frac{re^t+c_1}{2\sqrt{c_2}}\right)$$

where $\arg\tanh(u) = (1/2)\log((u+1)/(u-1)), |u| > 1$. The second term writes

$$\begin{aligned}\frac{Z_t}{Z_t-1} &= \frac{4re^t}{4c_2+4re^t-(re^t+c_1)^2} = \frac{4re^t}{4c_2-c_1^2+(c_1-2)^2-(re^t+c_1-2)^2}\\ &= \frac{re^t}{c_2+1-c_1-\left(\dfrac{re^t+c_1-2}{2}\right)^2} = \frac{1}{c_2+1-c_1}\frac{re^t}{1-\left(\dfrac{re^t+c_1-2}{2\sqrt{c_2+1-c_1}}\right)^2}\\ &= \frac{2}{\sqrt{c_2+1-c_1}}\frac{(r/2\sqrt{c_2+1-c_1})e^t}{1-\left(\dfrac{re^t+c_1-2}{2\sqrt{c_2+1-c_1}}\right)^2}\end{aligned}$$

Observe that $2-c_1-re^t > 2-c_1-re^{t_0} = 2(1-\theta(1+\lambda) \geq 0$. Thus, if $\theta(1+\lambda) \neq 1$

$$\frac{1-\theta(1+\lambda)}{2}\int\frac{Z_s}{Z_s-1}ds = \arg\tanh\left(\frac{2-c_1-re^t}{2\sqrt{c_2+1-c_1}}\right)$$

Thus, if $\lambda \neq 1\,(\theta \leq 1/2 < 1/(\lambda+1))$,

$$K_t = C(1-Z_t)^{1/2}\left(\frac{re^t + c_1 + 2\sqrt{c_2}}{re^t + c_1 - 2\sqrt{c_2}}\right)^{1/2}\left(\frac{2-c_1-2c_3-re^t}{2-c_1+2c_3-re^t}\right)^{1/2}$$

where $c_3 := \sqrt{c_2+1-c_1} = 1-\theta(\lambda+1)$. Note that for $\lambda = 1, \theta = 1/2,\ c_1 = 1, c_2 = 0, c_3 = 0$ and

$$K_t = C\frac{1-re^t}{1+re^t}, \qquad t < t_0 = -\ln r.$$

*The case $\theta = 1/2, \lambda \neq 1$: free martingales polynomials*: one has

$$c_1 = 1,\ c_2 = \frac{(1-\lambda)^2}{4},\ c_3 = \sqrt{c_2} = \frac{1-\lambda}{2},\ Z_t = \frac{4re^t}{(re^t+1)^2-(1-\lambda)^2}$$
$$c_1 + 2\sqrt{c_2} = 2(1+c_3) - c_1 = 2-\lambda,\ c_1 - 2\sqrt{c_2} = 2(1-c_3) - c_1 = \lambda.$$
$$1 - Z_t = \frac{(re^t-1)^2-(1-\lambda)^2}{(re^t+1)^2-(1-\lambda)^2} = \frac{(re^t+\lambda-2)(re^t-\lambda)}{(re^t+2-\lambda)(re^t+\lambda)}$$

Thus, for $t < -\ln(r/\lambda)$,

$$K_t = C\frac{\lambda - re^t}{\lambda + re^t}$$

so that

$$\begin{aligned}
R_t &= C\frac{\lambda - re^t}{\lambda + re^t}\left(P - \frac{4re^t}{(re^t+1)^2-(1-\lambda)^2}J_t\right)^{-1} \\
&= C(\lambda - re^t)(2-\lambda+re^t)(\lambda(2-\lambda)P + (re^t)^2P - 2re^t(2J_t - P))^{-1} \\
&= \frac{C(\lambda - re^t)(2-\lambda+re^t)}{\lambda(2-\lambda)}\left(P - \frac{2re^t}{\sqrt{\lambda(2-\lambda)}}\frac{(2J_t-P)}{\sqrt{\lambda(2-\lambda)}} + \frac{(re^t)^2}{\lambda(2-\lambda)}P\right)^{-1} \\
&= C\left(1 - 2\frac{(1-\lambda)}{\sqrt{\lambda(2-\lambda)}}\frac{re^t}{\sqrt{\lambda(2-\lambda)}} - \frac{(re^t)^2}{\lambda(2-\lambda)}\right)\left(P - \frac{2re^t}{\sqrt{\lambda(2-\lambda)}}\frac{(2J_t-P)}{\sqrt{\lambda(2-\lambda)}} + \frac{(re^t)^2}{\lambda(2-\lambda)}P\right)^{-1}
\end{aligned}$$

is a free martingale with respect to the natural filtration $\mathscr{J}_t$. Besides, since $\lambda \in ]0,1]$, then $\lambda \leq \sqrt{\lambda(2-\lambda)}$, hence $(re^t)/(\sqrt{\lambda(2-\lambda)}) < 1$ for all $t < -\ln(r/\lambda)$. Now, let us consider the following generating function

$$g(u,x) = \frac{1-2au-u^2}{1-2xu+u^2}, \quad 0 < a, u < 1,\ |x| \leq 1.$$

It follows that

$$g(u,x) = U_0(x) + (U_1(x) - 2a)u + \sum_{n\geq 2}[U_n(x) - 2aU_{n-1}(x) - U_{n-2}(x)]u^n$$

Setting

$$u_{t,\lambda} := \frac{re^t}{\sqrt{\lambda(2-\lambda)}}, \quad t < t_0,$$

then

$$R_t = C[P + (x_{t,\lambda} - 2a(\lambda)P)u_{t,\lambda} + \sum_{n\geq 2}[U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})]u_{t,\lambda}^n$$

Setting $U_{-1} = U_{-2} = 0$, it can be written as

$$R_t = C\sum_{n\geq 0}[U_n(x_{t,\lambda}) - 2a(\lambda)U_{n-1}(x_{t,\lambda}) - U_{n-2}(x_{t,\lambda})]u^n_{t,\lambda}$$

*Remark. The case* $\lambda = 1$.
$c_1 = 4\theta(1-\theta)$, $c_2 = 0$ and $Z_t$ writes

$$Z_t = \frac{4re^t}{(re^t + 4\theta(1-\theta))^2}$$

Moreover, $c_3 = \sqrt{1-c_1} = (1-2\theta)$, $2 - 2c_3 - c_1 = 4\theta^2$, $2 + 2c_3 - c_1 = 4(1-\theta)^2$. $K_t$ then writes

$$K_t = \frac{\sqrt{(re^t + 4\theta(1-\theta))^2 - 4re^t}}{re^t + 4\theta(1-\theta)}\sqrt{\frac{4\theta^2 - re^t}{4(1-\theta)^2 - re^t}}$$

## 4. ONE-PARAMETER MEASURES FAMILY AND ORTHOGONAL POLYNOMIALS

Let $\mu$ be a measure on the real line which is not supported by a finite set. Assume that $\mu$ has finite moments of all orders. Applying the Gram-Schmidt orthogonolization method to the basis $(1, x, x^2, \dots)$, there exist a unique family of monic orthogonal polynomials with respect to $\mu$, say $(P_n)_{n\geq 0}$. These polynomials satisfy the three-terms recurrence relation

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \quad n \geq 0, P_{-1} := 0.$$

where $\alpha_n \in \mathbb{R}$, $w_n > 0$. $(\alpha_n, \omega_n)_{n\geq 0}$ are called the Jacobi-Szegö parameters of $\mu$. It is known that $\mu$ is symmetric if and only if $\alpha_n = 0$, $n \geq 0$. Another way to derive the family $(P_n)_n$ is the multiplicative renormalization method ([3],[4],[5], [6]) that we shall recall here : a nice function $(u, x) \mapsto \psi(u, x)$ is a generating function for the measure $\mu$ if $\psi$ has the expansion

$$\psi(u, x) = \sum_{n\geq 0} c_n P_n(x) u^n, \qquad c_n \in \mathbb{R}$$

where $P_n$ are orthogonal with respect to $\mu$. Of course, there is more than one generating function corresponding to a given measure and in order to claim whether a function is a generating function or not, authors in [3] provided a necessary and sufficient condition. For a particular form of $\psi$ which fits our need, their result is formulated as follows:

**Theorem 4.1.** *Define*

$$\theta(u) := \int_{\mathbb{R}} \frac{1}{1-ux}\mu(dx), \quad \theta(u,v) := \int_{\mathbb{R}} \frac{1}{(1-ux)(1-vx)}\mu(dx).$$

*Let* $\rho$ *analytic around* $0$ *such that* $\rho(0) = 0$ *and* $\rho'(0) \neq 0$. *Then*

$$\psi(u,x) := \frac{(1-\rho(u)x)^{-1}}{\theta(\rho(u))} \tag{2}$$

*is a generating function for* $\mu$ *if and only if*

$$\Theta_\rho(u,v) := \frac{\theta(\rho(u),\rho(v))}{\theta(\rho(u))\theta(\rho(v))}$$

*is a function of* $uv$.

We will apply this result to the measures family $\nu_\lambda, \lambda \in ]0,1]$ which is the image of

$$\mu_{\lambda,1/2} = \frac{1}{\pi\lambda}\frac{\sqrt{(x_+ - x)(x-x_-)}}{x(1-x)}\mathbf{1}_{[x_-,x_+]}(x)dx, \quad x_\pm = \frac{(\sqrt{\lambda}\pm\sqrt{2-\lambda})^2}{4}$$

by the map

$$x \mapsto \frac{2x-1}{\sqrt{\lambda(2-\lambda)}}$$

Then,

$$\nu_\lambda(dx) = \frac{(2-\lambda)}{\pi}\frac{\sqrt{1-x^2}}{1-\lambda(2-\lambda)x^2}\mathbf{1}_{[-1,1]}(x)dx$$

Our scheme is the almost the same used in [9] except the computation of $\theta(u)$ which follows easily from $G_{\mu_{\lambda,1/2}}$. More precisely, authors considered the one-parameter measures family

$$\mu_a(dx) = \frac{a\sqrt{1-x^2}}{a^2+(1-2a)x^2}\mathbf{1}_{]-1,1[}dx, \quad a>0.$$

It is forward that $\mu_{1/(2-\lambda)} = \nu_\lambda$ almost everywhere for $0<\lambda\leq 1 \Leftrightarrow 1/2 < a \leq 1$.

**Proposition 4.1.**

$$\theta(u) = \theta_\lambda(u) = \frac{2-\lambda}{1-\lambda+\sqrt{1-u^2}}, \quad |u|<1$$

Using

$$\frac{1}{(1-ux)(1-vx)} = \frac{1}{u-v}\left(\frac{u}{1-ux} - \frac{v}{1-vx}\right)$$

it follows that $\theta(u,v) = (u\theta(u) - v\theta(v))/(u-v)$ from which we deduce

**Corollary 4.1.**

$$\theta(u,v) = \theta_\lambda(u,v) = \frac{1}{2-\lambda}\left[1-\lambda+\frac{u+v}{u\sqrt{1-v^2}+v\sqrt{1-u^2}}\right]$$

*Proof:* from the definition of $\nu_\lambda$, one writes for $0<u<\lambda(2-\lambda)\leq 1$:

$$\int_{\mathbb{R}}\frac{1}{1-ux}\nu_\lambda(dx) = \int_{\mathbb{R}}\frac{1}{1-u\dfrac{2x-1}{\sqrt{\lambda(2-\lambda)}}}\mu_{\lambda,1/2}(dx) = \frac{\sqrt{\lambda(2-\lambda)}}{2u}G_{\mu_{\lambda,1/2}}\left(\frac{\sqrt{\lambda(2-\lambda)}+u}{2u}\right)$$

The result follows from

$$G_{\mu_{\lambda,1/2}}(z) = \frac{(1-\lambda)(2z-1)-\sqrt{4z^2-4z+(1-\lambda)^2}}{2\lambda z(1-z)}, \quad z\in\mathbb{C}\setminus[0,1] \qquad \blacksquare$$

Let $\rho(u) = 2u/(1+u^2)$, then

$$\frac{\rho(u)+\rho(v)}{\rho(u)\sqrt{1-\rho^2(v)}+\rho(v)\sqrt{1-\rho^2(u)}} = \frac{1+uv}{1-uv}$$

so that Theorem 4.1 applies and claims that

$$\psi_\lambda(u,x) = \frac{1-\lambda/(2-\lambda)u^2}{1-2ux+u^2}$$

is a generating function for $\nu_\lambda$ corresponding to the polynomials

$$Q_n^\lambda(x) = U_n(x) - \frac{\lambda}{2-\lambda}U_{n-2}(x),, \quad n \geq 0,\, U_{-1} = U_{-2} := 0.$$

Using the recurrence relation

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \quad U_{-1} := 0, \tag{3}$$

These polynomials satisfy

$$\begin{aligned} 2xQ_0^\lambda(x) &= Q_1^\lambda(x) \\ 2xQ_1^\lambda(x) &= Q_2^\lambda(x) + \left(1 + \frac{\lambda}{2-\lambda}\right) Q_0^\lambda(x) \\ 2xQ_n^\lambda(x) &= Q_{n+1}^\lambda(x) + Q_{n-1}^\lambda(x),\, n \geq 2. \end{aligned}$$

Setting $Q_{-1}^\lambda := 0$ and since the coefficient of the leading power in $Q_n^\lambda(x)$ is $2^n$, then one deduces that the Jacobi-Szegö parameters are given by : $\alpha_n = 0, n \geq 0, w_1 = 1/(2(2-\lambda)), w_n = 1/4, n \geq 2$.

*Remark.* In [10], authors characterize the absolutely continuous measures for which the multiplicative renormalization method is applicable with the generating function given by (2). They derived a two-parameters densities family written as

$$f(x) = \frac{c\sqrt{1-x^2}}{\pi[b^2 + c^2 - 2b(1-c)x + (1-2c)x^2]}\mathbf{1}_{[-1,1]}(x), \quad |b| < 1-c,\, 0 < c \leq 1.$$

These densities fit the image of absolutely continuous part of $\mu_{\lambda,\theta}$ by the map

$$u = \frac{2x-s}{d} \in [-1,1]$$

with $d = d(\lambda,\theta) = x_+ - x_- = 4\theta\sqrt{\lambda(1-\theta)(1-\lambda\theta)}$, , $s = s(\lambda,\theta) = x_+ + x_- = 2\theta(1 + \lambda - 2\lambda\theta)$. One gets

$$\nu_{\lambda,\theta}(dx) = \frac{d^2}{2\pi\lambda\theta}\frac{\sqrt{1-x^2}}{s(2-s) + 2d(1-s)x - d^2x^2}dx$$

which provides the following relations

$$c = \frac{1}{2(1-\lambda\theta)}, \quad b = \sqrt{\frac{\lambda}{(1-\theta)(1-\lambda\theta)}}(2\theta - 1) \tag{4}$$

As a result, one can derive the correponding orthogonal polynomials for $\lambda \in ]0,1], \theta \leq 1/(\lambda+1)$ from the generating function ([10]):

$$\phi(u,x) = \frac{1 - 2bu + (1-2c)u^2}{1 - 2ux + u^2}. \tag{5}$$

## 5. MORE ORTHOGONAL POLYNOMIALS

Consider the polynomials $P_n^\lambda$ defined by

$$P_n^\lambda(x) = U_n(x) - 2a(\lambda)U_{n-1}(x) - U_{n-2}(x), \quad U_{-1} = U_{-2} := 0$$

with generating function

$$g(u,x) = \frac{1-2a(\lambda)u - u^2}{1-2xu+u^2}, \quad a(\lambda) = \frac{1-\lambda}{\lambda(2-\lambda)}, \, 0 < u < 1.$$

The $P_n^\lambda$'s appear in [2] as a limiting case of the $q$-Pollaczek polynomials. The coefficient of the highest monomial is equal to $2^n$. Using (3), one deduces that

$$\begin{aligned} 2[x-a(\lambda)]P_0^\lambda(x) &= P_1^\lambda(x) \\ 2xP_1^\lambda(x) &= P_2^\lambda(x) + 2P_0^\lambda(x) \\ 2xP_n^\lambda(x) &= P_{n+1}^\lambda(x) + P_{n-1}^\lambda(x), \quad n \geq 2. \end{aligned}$$

Thus the Jacobi-Szegö parameters are given by $\alpha_0 = a(\lambda)$ and $\alpha_n = 0$ for all $n \geq 1$ and $\omega_1 = 1/2$, $\omega_n = 1/4$, $n \geq 2$ ($P_{-1}^\lambda = 0$).
One can use Theorem 4.1 to determine the probability measure, $\xi_\lambda$, with respect to which the $P_n^\lambda$s are orthogonal. Since $\alpha_0 \neq 0$, then $\xi_\lambda$ is not symmetric. Indeed, keeping the same function $\rho$ previously defined, then the function $\theta$ must be equal to

$$\theta(\rho(u)) = \frac{1+u^2}{1-2a(\lambda)-u^2}$$

so that

$$\theta(u) = \frac{1}{\sqrt{1-u^2} - a(\lambda)u}$$

From the definition of $\theta$, one deduces that

$$G_{\xi_\lambda}(u) := \int_{\mathbb{R}} \frac{1}{u-x}\xi_\lambda(dx) = \frac{1}{u}\theta\left(\frac{1}{u}\right) = \frac{\sqrt{u^2-1}+a(\lambda)}{u^2-(1+a^2(\lambda))}$$

for $|u| > 1$, $u \neq \pm\sqrt{1+a(\lambda)^2}$. Thus, $\xi_\lambda$ has two atoms $a_\pm$ at $\pm\sqrt{a^2(\lambda)+1}$ and an absolutely continuous part given by

$$a_\pm = -\lim_{y\to 0^+} y\Im G_{\xi_\lambda}(\pm\sqrt{a^2(\lambda)+1}+iy), \quad g(x) = -\frac{1}{\pi}\lim_{y\to 0^+}\Im G_{\xi_\lambda}(x+iy)$$

Using that the Cauchy transform maps $\mathbb{C}^+$ to $\mathbb{C}^-$, one finally gets

$$\xi_\lambda(dx) = \frac{a(\lambda)}{\sqrt{a^2(\lambda)+1}}\delta_{\sqrt{a^2(\lambda)+1}}(dx) + \frac{1}{\pi}\frac{\sqrt{1-x^2}}{a^2(\lambda)+1-x^2}\mathbf{1}_{|x|<1}dx$$

*Remark.* To see that this defines a probability measure for $\lambda \neq 1$, it suffices to write

$$\begin{aligned} \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{a^2(\lambda)+1-x^2}dx &= \frac{1}{\pi}\int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}(a^2(\lambda)+1-x)}dx \\ &= \frac{1}{2(a^2(\lambda)+1)}{}_2F_1\left(1,\frac{1}{2},2;\frac{1}{a^2(\lambda)+1}\right) \end{aligned}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function given by

$${}_2F_1(e,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-e}dx,\ \Re(b)\wedge\Re(c-b)>0$$

for $|u|<1$. Then, one uses the identity

$${}_2F_1(1,b,2;z) = \frac{1-(1-z)^{1-b}}{(1-b)z}$$

to get

$$\frac{1}{\pi}\int_{-1}^1 \frac{\sqrt{1-x^2}}{a^2(\lambda)+1-x^2}dx = 1 - \frac{a(\lambda)}{\sqrt{a^2(\lambda)+1}}$$

## 6. One mode Interacting Fock space

In the sequel, we give a realization of $\nu_{\lambda,\theta}$, image of the spectral measure $\mu_{\lambda,\theta}$ for $\lambda \in ]0,1], \theta \leq 1/(\lambda+1)$ so that the support is $[-1,1]$. In the quantum scope, it is known as the quantum decomposition of $\nu_{\lambda,\theta}$. We only need the Jacobi-Szegö parameters in order to apply Accardi-Bozejko Theorem ([1]). We first write down from the generating function (5) the orthogonal polynomials (see [10]) corresponding to $\nu_{\lambda,\theta}$:

$$Q_n^{\lambda,\theta} = U_n - 2bU_{n-1} + (1-2c)U_{n-2}, \quad U_{-1}=U_{-2}=0,$$

where $b=b(\lambda,\theta), c=c(\lambda,\theta)$ are given by (4). It follows that $\alpha_0 = b, \alpha_n = 0$ for $n\geq 1$ and $\omega_1 = c/2, \omega_n = 1/4$ for $n \geq 1$. In order to use Accardi-Bozejko Theorem ([1]), we shall introduce the so-called *one-mode interacting Fock space*: let $\mathcal{H}$ be a one dimensional separable complex Hilbert space $\sim \mathbb{C}$. Then the $n$-th tensor product $\mathcal{H}^{\otimes n}$ is one dimensional: indeed $z_1\otimes\cdots\otimes z_n = (z_1\ldots z_n)1\otimes\cdots\otimes 1 \in \mathbb{C}\Phi_n$. The one-mode interacting Fock space associated to $\nu_{\lambda,\theta}$ is defined as $\Gamma(\mathbb{C}\Phi_n,(\lambda_n))$ as the infinite orhogonal sum of $\mathbb{C}\Phi_n$ equipped with the weighted scalar product

$$(z_1\Phi_n, z_2\Phi_n) := \lambda_n\overline{z_1}z_2, \quad z_1,z_2\in\mathbb{C},$$

where $\lambda_n = \omega_1\ldots\omega_n$. Then $\nu_{\lambda,\theta}$ is the vacuum distribution (in the vacuum state $\Phi_1$) of any extension of the operator $a^+ + a + \alpha_N$ where

$$\begin{aligned}
a^+\Phi_n &= \Phi_{n+1} \quad \text{(creation operator)}\\
a\Phi_{n+1} &= \omega_{n+1}\Phi_n = \frac{\lambda_{n+1}}{\lambda_n}\Phi_n,\ a\Phi_1 = 0, \quad \text{(annihilation operator)}\\
N\Phi_n &= n\Phi_n \qquad \text{(Number operator)}, \quad aa^+\Phi_n = \frac{\lambda_{n+1}}{\lambda_n}\Phi_n,
\end{aligned}$$

and $\alpha_N$ is defined by the spectral Theorem, that is $\alpha_N\Phi_n = \alpha_n\Phi_n$.

*Remark.* The concept of one mode interacting Fock space (IFS) is purely algebraic as the reader can see from [1] and is fully characterized by both the commutation relations between creation and annihilation operators and $a\Phi_1 = 0$. The most important feature of Accardi-Bozejko Theorem is illustrated in the *canonical* isomorphism between one mode IFS and the $L^2$-space of a given measure $\mu$ of all order moments. It is noteworthy that only the $\omega_n$s are involved in the commutation relations (thus in both one mode IFS and $L^2(\mu)$) while the $\alpha_n$s reflect only the symmetry of $\mu$.

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## References

[1] *L. Accardi, M. Bozejko.* Interacting Fock space and Gausssianization of probability measures. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.1.* **4**. 1998, 663-670.

[2] *W. A. Al-Salam, T. S. Chihara.* $q$-Pollaczek polynomials and a conjecture of Andrews and Askey. *SIAM J. Math. Anal.* **18** (1987), no. 1, 228–242.

[3] *N. Asai, I. Kubo, H. H. Kuo.* Multiplicative renormalization and generating function I. *Taiwanese J. Math. 7.* **1**. 2003, 89-101.

[4] *N. Asai, I. Kubo, H. H. Kuo.* Renormalization, orthogonalization and generating function. *Quantum Information V.* 2006, 49-55. World Sci. Publ. Hackensack, NJ.

[5] *N. Asai, I. Kubo, H. H. Kuo.*Multiplicative renormalization and generating function II. *Taiwanese J. Math. 8.* **4**. 2004, 593-628.

[6] *N. Asai, I. Kubo, H. H. Kuo.* Generating function method for orthogonal polynomials and Jacobi-Szegö parameters. *Probab. Math. Statist. 23.* **2**, 2003, 273-291. Acta Univ. Wratislav. No 2593.

[7] *P. Biane.* Free Brownian motion, free stochastic calculus and random matrices. *Fie. Inst. Comm.* **12**, Amer. Math. Soc. Providence, RI, 1997, 1-19.

[8] *N. Demni.* Free Jacobi process. To appear in *J. Theo. Proba.*

[9] *I. Kubo, H. H. Kuo, S. Namli.* Interpolation of Chebyshev polynomials and interacting Fock spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9.* **3**. 2006, 361-371.

[10] *I. Kubo, H. H. Kuo, S. Namli.* The class of measures applicable to the renormalization method for $(1-x)^{-1}$. Preprint 2007.

[11] *M. L. Silverstein.* Orthogonal polynomial martingales on spheres. *Sém. Probab. XX.* 1984/85, 419-422. Lecture notes in Math., **1204**, Springer, Berlin, 1986.

[12] *R. Speicher.* Combinatorics of Free Probability Theory. *Lectures. I. H. P. Paris.* 1999.